Logic for Computer Science. Knowledge Representation and Reasoning.

Lecture Notes

for

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Inference and Theorem Proving in Propositional Calculus

- Tasks and Models of Automated Inference,
- Theorem Proving models,
- Some important Inference Rules,
- Theorems of Deduction: 1 and 2,
- Models of Theorem Proving,
- Examples of Proofs,
- The Resolution Method,
- The Dual Resolution Method,
- Logical Derivation,
- The Semantic Tableau Method,
- Constructive Theorem Proving: The Fitch System,
- Example: The Unicorn,
- Looking for Models: Towards SAT.

Logic for KRR — Tasks and Tools

- Theorem Proving — Verification of Logical Consequence:

$$\Delta \models H;$$

- Method of Theorem Proving: Automated Inference —- Derivation:

$$\Delta \vdash H;$$

- SAT (checking for models) — satisfiability:

$$\models_I H \quad \text{(if such I exists)};$$

- un-SAT verification — unsatisfiability:

$$\not\models_I H \quad \text{(for any I)};$$

- Tautology verification (completeness):

$$\models H$$

- Unsatisfiability verification

$$\not\models H$$

Two principal issues:

- valid inference rules — checking:

$$(\Delta \vdash H) \quad \longrightarrow \quad (\Delta \models H)$$

- complete inference rules — checking:

$$(\Delta \models H) \quad \longrightarrow \quad (\Delta \vdash H)$$

Two Possible Fundamental Approaches: Checking of Interpretations versus Logical Inference

Two basic approaches – reasoning paradigms:

- systematic evaluation of possible interpretations — the 0-1 method; problem — combinatorial explosion; for n propositional variables we have 2^n interpretations!
- logical inference — derivation — with rules preserving logical consequence.

Notation: formula H (a Hypothesis) is derivable from Δ (a Knowledge Base; a set of domain axioms):

$$\Delta \vdash H$$

This means that there exists a sequence of applications of inference rules, such that H is *mechanically* derived from Δ.

Two principal issues in logical knowledge-based systems:

$$\Delta \vdash H \quad \text{versus} \quad \Delta \models H$$

i.e.

- is the derived formula valid?
- can any valid formula be derived?

An example derivation - for intuition

Just for intuition, let us consider an example of constructive proof by linear derivation:

$$\phi = (p \Rightarrow q) \wedge (r \Rightarrow s),$$

$$\varphi = (p \wedge r) \Rightarrow (q \wedge s).$$

This time we perform derivation of φ from ϕ:

$$\phi \vdash \varphi$$

A rough outline of derivation steps:

1.	$p \Rightarrow q$	by assumption;
2.	$r \Rightarrow s$	by assumption;
3.	$\underline{p \wedge r}$	we introduce an assumption;
4.	p	elimination of conjunction from (3);
5.	q	Modus Ponens (1) and (4);
6.	r	elimination of conjunction from (3;)
7.	s	Modus Ponens (2) and (6);
8.	$q \wedge s$	conjunction introduction from (5) and (7);
9.	$(p \wedge r) \vdash (q \wedge s)$	the derivation based on assumption (3);
10.	$(p \wedge r) \Rightarrow (q \wedge s)$	implication introduction based on (9)

Obviously, there is also:

$$\phi \models \varphi$$

But why?

Some more important inference rules !?! !?!

- $\frac{\alpha}{\alpha \vee \beta}$ — Disjunction Introduction,
- $\frac{\alpha,\ \beta}{\alpha \wedge \beta}$ — Conjunction Introduction,
- $\frac{\alpha \wedge \beta}{\alpha}$ — Conjunction Elimination,
- $\frac{\alpha,\ \alpha \Rightarrow \beta}{\beta}$ — Modus Ponens (modus ponendo ponens); implication elimination (EI),
- $\frac{\alpha \Rightarrow \beta,\ \neg\beta}{\neg\alpha}$ — Modus Tollens (modus tollendo tollens),
- $\frac{\alpha \vee \beta,\ \neg\alpha}{\beta}$ — Modus Tollendo Ponens,
- $\frac{\alpha \bigoplus \beta,\ \alpha}{\neg\beta}$ — Modus Ponendo Tollens,
- $\frac{\alpha \Rightarrow \beta,\ \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma}$ — Transitivity Rule,
- $\frac{\alpha \vee \gamma,\ \neg\gamma \vee \beta}{\alpha \vee \beta}$ — **Resolution Rule**,
- $\frac{\alpha \wedge \gamma;\ \neg\gamma \wedge \beta}{\alpha \wedge \beta}$ — **Dual Resolution Rule**; (backward) dual resolution (works backwards); also termed *consolution*,
- $\frac{\alpha \Rightarrow \beta,\ \gamma \Rightarrow \delta}{(\alpha \vee \gamma) \Rightarrow (\beta \vee \delta)}$ — Constructive Dilemma I,
- $\frac{\alpha \Rightarrow \beta,\ \gamma \Rightarrow \delta}{(\alpha \wedge \gamma) \Rightarrow (\beta \wedge \delta)}$ — Constructive Dilemma II.

The Deduction Theorems

Theorem 1 *Let $\Delta_1, \Delta_2, \ldots \Delta_n$ and Ω are logical formulas. Ω is their logical consequence iff $\Delta_1 \wedge \Delta_2 \wedge \ldots \Delta_n \Rightarrow \Omega$ is a tautology.*

Theorem 2 *Let $\Delta_1, \Delta_2, \ldots \Delta_n$ and Ω are logical formulas. Ω is their logical consequence iff $\Delta_1 \wedge \Delta_2 \wedge \ldots \Delta_n \wedge \neg\Omega$ is invalid (false under any interpretation).*

Theorem proving: having $\Delta_1, \Delta_2, \ldots \Delta_n$ assumed to be true show that so is Ω. Hence:

$$\Delta_1 \wedge \Delta_2 \wedge \ldots \Delta_n \models \Omega$$

Basic methods for theorem proving:

- evaluation of all possible interpretations (the 0-1 method),
- direct proof (forward chaining) – derivation of Ω from initial axioms,
- search for proof (backward chaining) – search for derivation of Ω from initial axioms; Backtracking Search,
- proving tautology – from the Deduction Theorem 1 we prove that $\Delta_1 \wedge \Delta_2 \wedge \ldots \Delta_n \Rightarrow \Omega$ is a tautology,
- indirect proof – through constraposition: $\neg\Omega \Rightarrow \neg(\Delta_1 \wedge \Delta_2 \wedge \ldots \Delta_n)$.
- Reductio ad Absurdum; basing on Deduction Theorem 2 we show that $\Delta_1 \wedge \Delta_2 \wedge \ldots \Delta_n \wedge \neg\Omega$. is unsatisfiable

Examples

Direct proof: $(p \Rightarrow r) \wedge (q \Rightarrow s) \wedge (\neg r \vee \neg s) \models (\neg p \vee \neg q)$:

1. $p \Rightarrow r$ assumption,
2. $q \Rightarrow s$ assumption,
3. $\neg r \vee \neg s$ assumption,
4. $s \Rightarrow \neg r$ implication reconstruction; through equivalence to 3,
5. $q \Rightarrow \neg r$ transitivity 2 and 4,
6. $\neg p \vee r$ EI from 1,
7. $\neg q \vee \neg r$ EI from 5
8. $\neg p \vee \neg q$ by resolution rule from 6 and 7.

Proving tautology: $[p \Rightarrow (q \Rightarrow r)] \models [q \Rightarrow (p \Rightarrow r)]$.
We transform the formula $[p \Rightarrow (q \Rightarrow r)] \Rightarrow [q \Rightarrow (p \Rightarrow r)]$ and through elimination of implications we obtain $\alpha \vee \neg\alpha$.

Indirect proof: $p \models \neg q \Rightarrow \neg(p \Rightarrow q)$

1. $\neg(\neg q \Rightarrow \neg(p \Rightarrow q))$ assumption (contraposition),
2. $\neg(q \vee \neg(p \Rightarrow q))$ EI,
3. $(\neg q \wedge (p \Rightarrow q))$ De Morgan rule,
4. $\neg q$ CE,
5. $p \Rightarrow q$ CE from 3,
6. $\neg p \vee q$ EI from 5,

7. $q \vee \neg p$ commutativity from 6,

8. $\neg p$ RR from 4 and 7.

Reductio ad Absurdum: $(p \vee q) \wedge \neg p \models q$

1. $p \vee q$ assumption,

2. $\neg p$ assumption,

3. $\neg q$ assumption (negation of the hypothesis),

4. q RR to 1 and 2

5. $\perp$ from 3 and 4.

Example: Logical Consequence

$$\frac{(p \Rightarrow q) \wedge (r \Rightarrow s)}{(p \vee r) \Rightarrow (q \vee s)}$$

Let us put:

$$\phi = (p \Rightarrow q) \wedge (r \Rightarrow s)$$

and

$$\varphi = (p \vee r) \Rightarrow (q \vee s),$$

So we have to check if:

$$\phi \models \varphi. \tag{1}$$

p	q	r	s	$p \Rightarrow q$	$r \Rightarrow s$	$(p \Rightarrow q) \wedge (r \Rightarrow s)$	$p \vee r$	$q \vee s$	$(p \vee r) \Rightarrow (q \vee s)$
0	0	0	0	1	1	**1**	0	0	**1**
0	0	0	1	1	1	**1**	0	1	**1**
0	0	1	0	1	0	**0**	1	0	**0**
0	0	1	1	1	1	**1**	1	1	**1**
0	1	0	0	1	1	**1**	0	1	**1**
0	1	0	1	1	1	**1**	0	1	**1**
0	1	1	0	1	0	**0**	1	1	**1**
0	1	1	1	1	1	**1**	1	1	**1**
1	0	0	0	0	1	**0**	1	0	**0**
1	0	0	1	0	1	**0**	1	1	**1**
1	0	1	0	0	0	**0**	1	0	**0**
1	0	1	1	0	1	**0**	1	1	**1**
1	1	0	0	1	1	**1**	1	1	**1**
1	1	0	1	1	1	**1**	1	1	**1**
1	1	1	0	1	0	**0**	1	1	**1**
1	1	1	1	1	1	**1**	1	1	**1**

From columns 7 and 10 we conclude that there is logical consequence (but no equivalence — 7, 10, 12 i 15).

The Resolution Method

1. Problem:

$$\Delta \models H$$

2. From Deduction Theorem 2:

$$\Delta \cup \neg H$$

   should be unsatisfiable.

3. Transform $\Delta \cup \neg H$ to CNF.

4. Using the RR derive an empty formula $\bot$.

Example:

1. Problem:

$$(p \Rightarrow q) \wedge (r \Rightarrow s) \models (p \vee r) \Rightarrow (q \vee s)$$

2. From Deduction Theorem 2 — show that:

$$[(p \Rightarrow q) \wedge (r \Rightarrow s)] \cup \neg[(p \vee r) \Rightarrow (q \vee s)]$$

   is unsatisfiable.

3. After transformation to CNF we have:

$$\{\neg p \vee q, \neg r \vee s, p \vee r, \neg q, \neg s\}$$

4. Derive $\bot$.

Dual Resolution Method

1. Problem:
$$\Delta \models H$$
2. From Deduction Theorem 1 show that:
$$\Delta \Rightarrow H$$
is a tautology.
3. Transform $\Delta \Rightarrow H$ to DNF.
4. Using the DRR derive an empty formula — the always true one $\top$.

Example:

1. Problem:
$$(p \Rightarrow q) \wedge (r \Rightarrow s) \models (p \vee r) \Rightarrow (q \vee s)$$
2. From Deduction Theorem 1 show that:
$$[(p \Rightarrow q) \wedge (r \Rightarrow s)] \Rightarrow [(p \vee r) \Rightarrow (q \vee s)]$$
is a tautology.
3. After transformation to DNF we have:
$$\{p \wedge \neg q; r \wedge \neg s; \neg p \wedge \neg r; q; s\}$$
4. Using the DRR derive an empty formula — the always true one $\top$.

Example of Resolution Derivation

A – signal from process,

P – signal added to a queue,

B – signal blocked by process,

D – signal received by process,

S – state of the process saved,

M – signal mask read,

H – signal management procedure activated,

N – procedure executed in normal mode,

R – process restart from context,

I – process must re-create context.

Rules — axiomatization:

$A \longrightarrow P$,

$P \wedge \neg B \longrightarrow D$,

$D \longrightarrow S \wedge M \wedge H$,

$H \wedge N \longrightarrow R$,

$H \wedge \neg R \longrightarrow I$,

Facts:

A, $\neg B$, $\neg R$.

Application of RR to CNF:

$$\{\neg A \vee P, \neg P \vee B \vee D, \neg D \vee S, \neg D \vee M, \neg D \vee H, \neg H \vee \neg N \vee R, \neg H \vee R \vee I, A, \neg B, \neg R\}$$

Conclusions

P, D, S, M, H, I, $\neg N$.

Inference step; derivation

Step of inference: single application of RR.

Example:

Application of RR:

$$\frac{\phi \vee \neg p, p \vee \psi}{\phi \vee \psi}$$

Notation: $\{\phi \vee \neg p, p \vee \psi\} \vdash_R \phi \vee \psi$

Definition 1 **Derivation** *A derivation of ϕ from Δ we call a sequence:*

$$\phi_1, \phi_2 \ldots \phi_k$$

such that:

- *formula ϕ_1 is derivable from Δ (in a single step):*

$$\Delta \vdash \phi_1,$$

- *every next formula is derivable from Δ and the earlier-derived formulas:*

$$\{\Delta, \phi 1, \phi_2, \ldots, \phi_i\} \vdash \phi_{i+1}$$

  for $i = 2, 3, \ldots, k-1$,

- *ϕ is the last formula:*

$$\phi = \phi_k$$

Notation: $\Delta \vdash \phi$, and ϕ is called *derivable from* Δ.

Set of Logical Consequences Cn

Definition 2 *Let Δ be set of formulas. The set of logical consequences is:*

$$Cn(\Delta) = \{\phi \colon \Delta \models \phi\}$$

where every ϕ contains (only) propositional symbols of Δ.

Lemma 1 **Properties of Cn** *There are:*

- $\Delta \subseteq Cn(\Delta)$,
- *monotonicity — if $\Delta_1 \subseteq \Delta_2$, then:*

  $$Cn(\Delta_1) \subseteq Cn(\Delta_2)$$

- $Cn(Cn(\Delta)) = Cn(\Delta)$ *(the so-called fixed point).*

Is the Fixed Point unique? Is it finitely defined ? Is it finite ?

Example: Consider the following set of formulas:

$$\Delta = \{\neg(\neg p \wedge \neg r), r \Rightarrow q, \neg q, p \Rightarrow t, \neg(t \wedge \neg s)\}$$

Show that:

$$\Delta \models s$$

The Semantic Tableau Method

Recall the notions of: an atom, a literal, a positive literal, a negative literal $\{p, \neg p\}$.

Recall that a formula $p \wedge \neg p$ is always false. Formla $p \vee \neg p$ is always true.

Assumptions:

- we consider satisfiability of a formula,
- the starting point is the formula in original form! (it is not necessary to transform it into the CNF/DNF),
- by analysis and decomposition we search for a model; no model means unsatisfiability,
- we develop a tree (or a table):
  - for conjunctive formals we develop a single branch (a linear form),
  - for disjunctive formulas we develop branches,
- existence of a pair of complementary literals closes a given branch (falsifies),
- lack of complementary literals — leads to a model (satisfiability),
- closing each branch means unsatisfiability of the original formula.

Example 1:

$$p \wedge (\neg q \vee \neg p)$$

Example 2:

$$(p \vee q) \wedge (\neg p \wedge \neg q)$$

Examples

Example 1:

$$p \wedge (\neg q \vee \neg p)$$

$$p, \neg q \vee \neg p$$

$$p, \neg q \qquad\qquad p, \neg p$$

Example 2:

$$(p \vee q) \wedge (\neg p \wedge \neg q)$$

$$p \vee q, \neg p \wedge \neg q$$

$$p \vee q, \neg p, \neg q$$

$$p, \neg p,\ \neg q \qquad\qquad q, \neg p, \neg q$$

Semantic Tableau Algorithm

Rules of transformation for conjunctive formulas (type α):

α	α_1	α_2
$\neg\neg A$	A	
$A_1 \wedge A_2$	A_1	A_2
$\neg(A_1 \vee A_2)$	$\neg A_1$	$\neg A_2$
$\neg(A_1 \Rightarrow A_2)$	A_1	$\neg A_2$
$A_1 \Leftrightarrow A_2$	$A_1 \Rightarrow A_2$	$A_2 \Rightarrow A_1$

Rules of transformation for disjunctive formulas (type β):

β	β_1	β_2
$B_1 \vee B_2$	B_1	B_2
$\neg(B_1 \wedge B_2)$	$\neg B_1$	$\neg B_2$
$B_1 \Rightarrow B_2)$	$\neg B_1$	B_2
$\neg(B_1 \Leftrightarrow B_2)$	$\neg(B_1 \Rightarrow B_2)$	$\neg(B_2 \Rightarrow B_1)$

An Algorithm for developing the Semantic Tree:

- The Root: the initial formula (in original form; WFF),
- U (for leaves) contains literals only:
  - $p, \neg p \in U$ — stop/falsification; *else*
  - stop/a model found,
- For a conjunctive formula $\alpha \in U$:

$$U' = (U - \{\alpha\}) \cup \{\alpha_1, \alpha_2\}$$

- For a disjuctive formula $\beta \in U$ we have branching:

$$U' = (U - \{\beta\}) \cup \{\beta_1\}$$

$$U'' = (U - \{\beta\}) \cup \{\beta_2\}$$

Example:

1. Problem:

$$(p \Rightarrow q) \wedge (r \Rightarrow s) \models (p \vee r) \Rightarrow (q \vee s)$$

2. Based on the Deduction Theorem (2), it should be shown that:

$$[(p \Rightarrow q) \wedge (r \Rightarrow s)] \cup \neg[(p \vee r) \Rightarrow (q \vee s)]$$

   is unsatisfiable.

3. *Transform to CNF. We have:*

$$\{\neg p \vee q, \neg r \vee s, p \vee r, \neg q, \neg s\}$$

4. *Using Resolution Rule derive an empty clause — always false.*

Problem: show that the following set of formulas is unsatisfiable with use of Semantic Tableau method.

$$[(p \Rightarrow q) \wedge (r \Rightarrow s)] \cup \neg[(p \vee r) \Rightarrow (q \vee s)]$$

In fact, we have a formula:

$$[(p \Rightarrow q) \wedge (r \Rightarrow s)] \wedge \neg[(p \vee r) \Rightarrow (q \vee s)]$$

Constructive Theorem Proving: The Fitch System

- AND Introduction (AI):

$$\frac{\phi_1, \ldots, \phi_n}{\phi_1 \wedge \ldots \wedge \phi_n}$$

- AND Elimination (AE):

$$\frac{\phi_1 \wedge \ldots \wedge \phi_n}{\phi_i}$$

- OR Introduction (OI):

$$\frac{\phi_i}{\phi_1 \vee \ldots \vee \phi_n}$$

- OR Elimination (OE):

$$\frac{\phi_1 \vee \ldots \vee \phi_n, \phi_1 \Rightarrow \psi, \ldots \phi_n \Rightarrow \psi}{\psi}$$

- Negation Introduction (NI):

$$\frac{\phi \Rightarrow \psi, \phi \Rightarrow \neg\psi}{\neg\phi}$$

- Negation Elimination (NE):

$$\frac{\neg\neg\phi}{\phi}$$

- Implication Introduction (II):

$$\frac{\phi \vdash \psi}{\phi \Rightarrow \psi}$$

- Implication Elimination (IE):

$$\frac{\phi, \phi \Rightarrow \psi}{\psi}$$

- Equivalence Introduction (EI),
- Equivalence Elimination (EE)

Example: Unicorn

Given the following Knowledge Base (KB):

- If the unicorn is mythical, then it is immortal
- If the unicorn is not mythical, then it is a mortal mammal
- If the unicorn is either immortal or a mammal, then it is horned
- The unicorn is magical if it is horned

answer the following questions:

- Is the unicorn mythical? (M)
- Is it magical? (G)
- Is it horned? (H)

In terms of logic:

$$\text{KB} \models G, H, M$$

$$\text{KB} \vdash G, H, M$$

Unicorn - Logical Model

Definition of propositional variables:

- M: The unicorn is mythical
- I: The unicorn is immortal
- L: The unicorn is mammal
- H: The unicorn is horned
- G: The unicorn is magical

Building a **Logical Model** for the puzzle:

- If the unicorn is mythical, then it is immortal:

$$M \longrightarrow I$$

- If the unicorn is not mythical, then it is a mortal mammal:

$$\neg M \longrightarrow (\neg I \wedge L)$$

- If the unicorn is either immortal or a mammal, then it is horned:

$$(I \vee L) \longrightarrow H$$

- The unicorn is magical if it is horned:

$$H \longrightarrow G$$

Resulting Boolean formula (the Knowledge Base):

$$(M \longrightarrow I) \wedge (\neg M \longrightarrow (\neg I \wedge L)) \wedge ((I \vee L) \longrightarrow H) \wedge (H \longrightarrow G)$$

A Solution: Formal Derivation of Logical Consequences

1. $(M \longrightarrow I) \equiv (\neg M \vee I)$
2. $(\neg M \longrightarrow (\neg I \wedge L)) \equiv (M \vee (\neg I \wedge L))$
3. $(M \vee (\neg I \wedge L)) \equiv ((M \vee \neg I) \wedge (M \vee L))$
4. $\neg M \vee I, M \vee L$
5. $I \vee L$
6. $I \vee L, (I \vee L) \longrightarrow H$
7. H
8. $H, H \longrightarrow G$
9. G

So we have:

$$\mathrm{KB} \vdash H \wedge G$$

Questions:

- What about M (mythical), I (immortal) and L (mammal)?
- What are the exact models? What combinations are admissible?
- How many models do we have?
- What is the CNF of the original formula?
- What is the DNF of the original formula?
- Resolution, Dual Resolution, Semantic Tableau, Fitch System,...
  Try each one; which one you prefer?