Logic for Computer Science. Knowledge Representation and Reasoning.

Lecture Notes

for

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Limitations of Propositional Calculus: Expressive Power

Consider the following examples in Natural Language (NL):

Adam is Bob's brother. Bob is Adam's brother.

If Adam is Bob's brother then Bob is Adam's brother.

If X is Y's brother then Y is X's brother.

If X is father of Y, and Y is father of Z, then X is grandfather of Z.

If block A is on block B, and block B is on block C, then A is above C.

Everything has it's price.

There is no free lunch. ("No free lunch" (NFL) theorem...)

Everybody loves someone. Everybody is loved by someone.

If everybody loves someone then anyone is loved by somebody.

If X is connected to Y, and Y is connected to Z, then X is connected to Z.

Every student of AGH is smart. Jan is a student of AGH. Jan is smart.

There exists a set of all sets.

The barber shaves anyone who does not shave himself.

Predicate Logic – New Opening: High Expressive Power. Objects, Variables, Relations, Constructions and Operations

New components:

- Constants — representation of individual (atomic) objects,
- Variables — symbols of unknown/universal objects
- Predicate symbols — names of relations among objects,
- Quantifiers — there exists/existential quantifier; for all/universal quantifier,
- Terms — objects of complex structure; connected atomic objects.

Logical connectives:

- negation,
- conjunction, disjunction,
- implication, equivalence.

Operations:

- Abstraction — from individual objects properties to universal properties,
- Specification — from universal properties to specific ones,
- Properties of Relations — e.g. symmetry, transitivity,
- Specification of Constraints — relations + variables + quantification
- Complex Logical formulas — with use of logical connectives,
- Complex Logical Inference — general rules, universal laws.

Basic limitation of Propositional Logic: no Universal Rules

Consider the following classical example:

Socrates is a man.
Every man is mortal.

Socrates is mortal.

```
man(plato).
man(socrates).
mortal(X):- man(X).
```

```
mother(eva,nadjed).
father(john,tom).
father(john,ted).
father(john,eva).
father(ted,jimmy).
man(tom).
man(ted).
woman(eva).

parent(X,Y):- father(X,Y).
parent(X,Y):- mother(X,Y).

brother(B,X):-
   parent(P,B),
   parent(P,X),
   man(B),
   B \= X.
```

```
uncle(U,X):-
   parent(P,X),
   brother(U,P).
```

Alphabet and Notation

Definition 1 *A relation R is any subset of Cartesian Product of some given sets:*

$$R \subseteq X_1 \times X_2 \times \ldots \times X_n$$

Relation is a set. Elements of any relations are tuples of the form $(x_1, x_2, \ldots, x_n)$.

Notation: $R(x_1, x_2, \ldots, x_n)$ is read: R holds for arguments $x_1, x_2, \ldots, x_n$.

Let there be given the following, pairwise disjoint four sets of symbols:

- C — a set of constant symbols (or constants, for short),
- V — a set of variable symbols (or variables, for short),
- F — a set of function (term) symbols,
- P — a set of relation (predicate) symbols.

Definition 2 **Terms:**

- **if c is a constant, $c \in C$, then $c \in TER$;**
- **if X is a variable, $X \in V$, then $X \in TER$;**
- **if f is an n-ary function symbol, $f \in F$, and $t_1, t_2, \ldots, t_n$ are terms, then $f(t_1, t_2, \ldots, t_n) \in TER$;**
- **all the elements of TER are generated only by applying the above rules.**

The number n is referred to as the arity of f. Notation:

$$f/n$$

Examples of Terms

Assume that $a, b, c \in C$, $X, Y, Z \in V$, $f, g \in F$, and arity of f and g is 1 and 2, respectively.

Then, all the following expressions are examples of terms:

- a, b, c;
- X, Y, Z;
- $f(a), f(b), f(c), f(X), f(Y), f(Z)$;
  $g(a, b), g(a, X), g(X, a), g(X, Y)$;
  $f(g(a, b)), g(X, f(X)), g(f(a), g(X, f(Z)))$.

The set of terms (even for one constant and functional symbol) is:

- infinite,
- countable.

Each term can be represented as a tree.

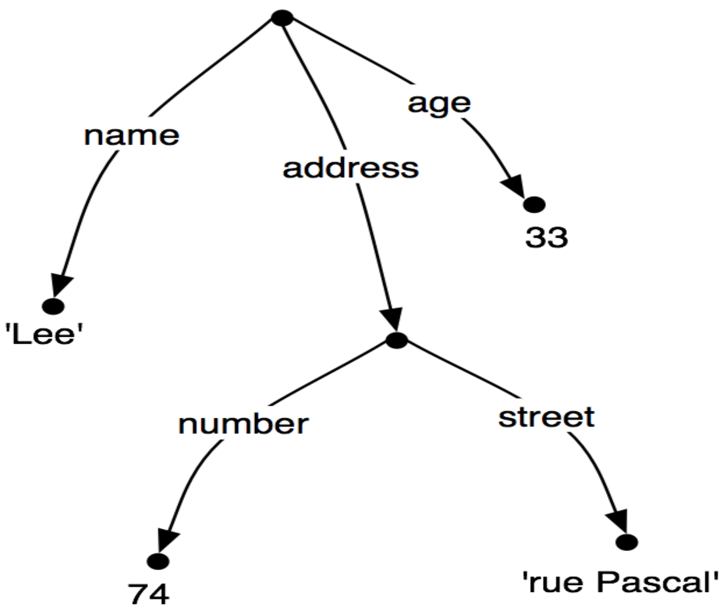


Figure 1: Visualization of the tree-like structure of a term

Example applications of Terms

Prolog:

```
book (book_title,
       author(first_name,last_name),
       publisher_name,
       year_of_publication
      )
```

XML:

```
<book>
      <book_title> Learning XML </book_title>
      <author>
            <first_name> Erik </first_name>
            <last_name>  Ray  </last_name>
      </author>
      <publisher_name> O'Reilly & Associates, Inc. </publisher
      <year_of_publication> 2003 </year_of_publication>
</book>
```

YAML:

```
book:
   title:      book_title
   author:     author_name
   publisher:  publisher_name
   year:       year_of_publication
```

Applications of Terms

LaTeX

$$\frac{\frac{x}{y}}{\sqrt{1+\frac{x}{y}}},$$

```
\frac{
      \frac{x}{y}
     }
     {
      \sqrt{1+\frac{x}{y}}
     }
```

Lists:

```
[red, green, blue, yellow]
[red|green, blue, yellow]
list(red,list(green,list(blue,list(yellow,nil))))
```

Trees:

```
tree (
      tree (left_left,  left_right),
      tree (right_left, right_right)
     )
```

Other: records, complex structures, natural numbers (Peano arithmetics),...

Formulas

Definition 3 *The set of Atomic Formulas* $ATOM$ *is defined as one satisfying the following conditions:*

- *if* p *is an* n*-ary predicate symbol,* $p \in P$*, and* $t_1, t_2, \ldots, t_n$ *are terms, then* $p(t_1, t_2, \ldots, t_n) \in ATOM$*;*
- *all the elements of* $ATOM$ *are generated by applying the above rule.*

The elements of $ATOM$ *are called* atomic formulae *or* atoms*, for short.*

Examples of atomic formulas:

- $p(a), p(b), q(a, a), q(a, c)$;
- $p(X), p(Y), q(X, X), q(X, Z)$;
- $p(f(a)), p(f(X)), q(f(g(a, b)), g(X, f(X))), q(g(f(a), g(X, f(Z))), a)$.

Terms vs. Atomic Formulas — what is the difference?

Definition 4 *Formulas:* FOR

- $ATOM \subseteq \mathrm{FOR}$*;*
- *if* Φ *is a formula,* $\Phi \in \mathrm{FOR}$*, then* $\neg(\Phi) \in \mathrm{FOR}$*;*
- *if* Φ *and* Ψ *are formulae,* $\Phi, \Psi \in \mathrm{FOR}$*, then* $(\Phi \wedge \Psi)$*,* $(\Phi \vee \Psi)$*,* $(\Phi \Rightarrow \Psi)$*,* $(\Phi \Leftrightarrow \Psi) \in \mathrm{FOR}$*;*
- *if* $\Phi \in \mathrm{FOR}$*,* X *denotes a variable, then* $\forall X(\Phi) \in \mathrm{FOR}$ *and* $\exists X(\Phi) \in \mathrm{FOR}$*;*
- *all the elements of* FOR *must be generated by applying the above rules.*

General note:

Although we can define formulas such as $\forall X: p$ or $\exists X: q$ it seems no to be rational; it is reasonable to qunatify over variables occurring in a formula, e.g. $\forall X \exists Y: p(X, Y)$

Notation:

- restricted general quantifier: $\forall X \in D_X$, $\forall_{X \in D_X}$,
- there exists exactly one element: $\exists ! X$,
- $\forall$ — general qunatifier (generalized conjunction); also: $\bigwedge$,
- $\exists$ — existential quantifier (generalized disjunction); also: $\bigvee$,

Generalization of conjunction:

$$\forall X: p(X) \quad \stackrel{?}{\equiv} \quad p(a) \wedge p(b) \wedge p(c) \wedge \ldots$$

Generalization of disjunction:

$$\exists X: p(X) \quad \stackrel{?}{\equiv} \quad p(a) \vee p(b) \vee p(c) \vee \ldots$$

Formulas (terms) with no variables: ground formulas/ground instances.

Free variable: X is a free variable in $p(X)$.

Bound variable: X is a bound variable in $\forall X: p(X)$.

We can construct formulas with free and bound variables...

The Roles of Variables. Free and Bound Variables

The role of variables is three-fold:

- names/references to unknown objects — they define boundings to quantifiers; free variables are not in use,
- placeholders — they keep place for (unknown) objects; arity!
- coreference constraints — they define coreferrence constraints (binding of occurrences; data carriers).

Occurrence of a variable in a formula can be:

- bound — within the scope of a quantifier,
- free — out of the scope of any quantifier,

A variable is bound in a formula iff all its occurrences are bound.

Definition 5 *Free variables in a formula:* $FV()$

- *if* $t \in V$ *then* $FV(t) = \{t\}$*;*
- *if* $t \in C$ *then* $FV(t) = \emptyset$*;*
- *if* $t = f(t_1, t_2, \ldots, t_n) \in \mathrm{TER}$ *then* $FV(t) = FV(t_1) \cup FV(t_2) \cup \ldots \cup FV(t_n)$*;*
- *if* $q = p(t_1, t_2, \ldots, t_n) \in ATOM$ *then* $FV(q) = FV(t_1) \cup FV(t_2) \cup \ldots \cup FV(t_n)$*;*
- $FV(\neg\Phi) = FV(\Phi)$*;*
- $FV(\Phi \diamond \Psi) = FV(\Phi) \cup FV(\Psi)$ *for any* $\diamond \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$*;*
- $FV(\nabla X(\Phi)) = FV(\Phi) \setminus \{X\}$ *for* $\nabla \in \{\forall, \exists\}$*.*

The Universe, Interpretation and Variable Assignment

In order to define the sematics we need:

- D — an non-empty set, the Universe,
- I — an Interpretation — a mapping of constants, function symbols, and predicate symbols into elements of D, functions over D and relations over D,
- v — Variable Assignment — assignment of elements of D (ground terms) to free variables.

Definition 6 *The Variable Assignment v:*

$$v\colon V \to D$$

may be defined over the elements of the universe (for simplicity)

Definition 7 *An interpretation I:*

- *for any constant $c \in C$, $I(c) \in D$;*
- *for any free occurrence of variable $X \in V$, $I(X) = v(X)$, where $v(X) \in D$;*
- *for any function symbol $f \in F$ of arity n, $I(f)$ is a function of the type*
$$I(f)\colon D^n \to D;$$
- *for any predicate symbol $p \in P$ of arity n, $I(p)$ is a relation such that*
$$I(p) \subseteq D^n;$$
- *for any term $t \in \mathrm{TER}$, such that $t = f(t_1, t_2, \ldots, t_n)$,*
$$I(t) = I(f)(I(t_1), I(t_2), \ldots, I(t_n)).$$

Semantics of the Predicate Calculus

Semantics = assignment of the meaning in the considered World; let:

- D — be the Universe, ,
- I — be the Interpretation, and
- v — be the Variable Assignment (or, for simplicity, we consider only closed formulas).

Definition 8 **Formulas satisfaction**

1. $\models_{I,v} p(t_1, t_2, \ldots, t_n)$ **iff (if and only if)** $(I(t_1), I(t_2), \ldots, I(t_n)) \in I(p)$ **(recall that** $I(X) = v(X)$ **for any free variable** $X \in VAR$**;**
2. $\models_{I,v} \neg\Phi$ **iff** $\not\models_{I,v} \Phi$**;**
3. $\models_{I,v} \Phi \wedge \Psi$ **iff both** $\models_{I,v} \Phi$ **and** $\models_{I,v} \Psi$**;**
4. $\models_{I,v} \Phi \vee \Psi$ **iff** $\models_{I,v} \Phi$ **or** $\models_{I,v} \Psi$**;**
5. $\models_{I,v} \Phi \Rightarrow \Psi$ **iff** $\not\models_{I,v} \Phi$ **or** $\models_{I,v} \Psi$**;**
6. $\models_{I,v} \Phi \Leftrightarrow \Psi$ **iff** $\models_{I,v} \Phi$ **and** $\models_{I,v} \Psi$**, or,** $\not\models_{I,v} \Phi$ **and** $\not\models_{I,v} \Psi$**;**
7. $\models_{I,v} \forall X \Phi$ **iff for any** $d \in D$ **and any variable assignment** u **such that** $u(X) = d$ **and** $u(Y) = v(Y)$ **for any** $Y \neq X$**, there is** $\models_{I,u} \Phi$**;**
8. $\models_{I,v} \exists X \Phi$ **iff there exists** $d \in D$ **such that for variable assignment** u **defined as** $u(X) = d$ **and** $u(Y) = v(Y)$ **for any** $Y \neq X$**, there is** $\models_{I,u} \Phi$**.**

Important Comments

For simplicity we consider closed formulas — no free variables are allowed.

If in a formula there are free variables, should they be considered universally quantified or existentially quantified? Or a combination of that? Example: $p(X) \vee \neg p(Y)$ is a tautology or not?

For convenience and for clarity, all the occurrences of variables are renamed in a consequent way so that no conflicts of variable names exist.

Definition 9 **Logical Consequence** *(case of closed formulas):*

A formula H is a logical consequence of set of formulas $\Delta_1, \Delta_2, \ldots, \Delta_n$ if and only if for any interpretation I (and universe D) satisfying $\Delta_1 \wedge \Delta_2 \wedge \ldots \wedge \Delta_n$, H is also satisfied under interpretation I (and universe D).

Example: How many possible interpretations can be defined for the following formulas:

- p,
- $p \wedge q$, $p \vee q$, $p \Rightarrow q$,
- $p(a)$,
- $p(f(a)))$,
- $\forall X: p(X)$,
- $\exists X: p(X)$.

The Herbrand Universe

Definition 10 *Herbrand Universe:*

Let $H_0 = C(\Delta)$, i.e. H_0 contains all the constants occurring in some set of formulas Δ (if $C(\Delta) = \emptyset$ then one defines H_0 in such a way that it contains a single arbitrary symbol, say $H_0 = \{c\}$).

Now, for $i = 0, 1, 2, \ldots,$ let $H_{i+1} = H_i \cup \{f(t_1, t_2, \ldots, t_n)\colon f \in F(\Delta)\ and\ t_1, t_2, \ldots, t_n \in H_i\}$ (where the arity of f is n). Then H_∞ is called the Herbrand Universe *of Δ.*

Definition 11 *The Herbrand Base:*

*Let Δ be a set of formulas and let **H** be the Herbrand Universe of Δ. A set $B_H = \{p(h_1, h_2, \ldots, h_n)\colon h_1, h_2, \ldots, h_n \in$ **H**$, p \in P(\Delta)\}$ (where the arity of p is n) is called the* Herbrand base *or the* atom set *of Δ.*

Definition 12 *The Herbrand Interpretation:*

*Let Δ be a set of formulas and let **H** be the Herbrand Universe of Δ. Any interpretation I_H is called a* Herbrand interpretation *(**H**-interpretation) if the following conditions are satisfied:*

- *for any constant $c \in$ **H**, $I_H(c) = c$;*
- *for any n-ary functional symbol $f \in F(\Delta)$, and any $h_1, h_2, \ldots, h_n \in$ **H**,*

$$I_H(f)\colon (h_1, h_2, \ldots, h_n) \to f(h_1, h_2, \ldots, h_n).$$

The Herbrand Universe - an Example

Consider the following example Block World:

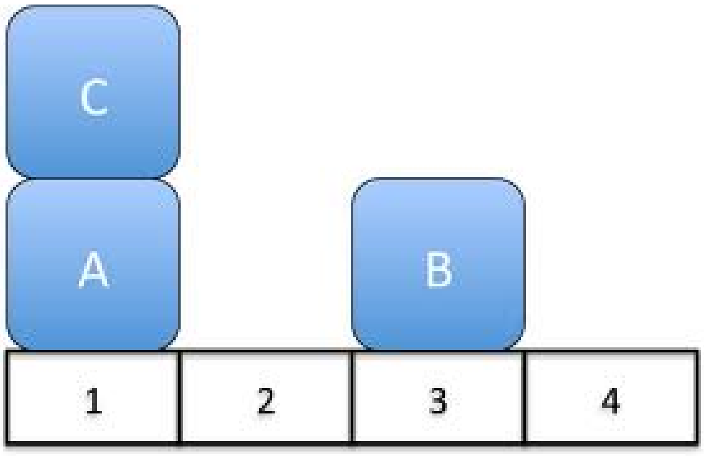


Figure 2: Block World Example

The Herbrand Universe is $\{a, b, c, 1, 2, 3, 4\}$; it is finite (no terms).

Let $\Delta = \{on/2\}$ – we have a single 2-argument predicate.
The Herbrand Base is:
$\{on(a, a), on(a, b), on(a, c), on(a, 1), on(a, 2), on(a, 3), on(a, 4)$
$\{on(b, a), on(b, b), on(b, c), on(b, 1), on(b, 2), on(b, 3), on(b, 4)$
...
$\{on(4, a), on(4, b), on(4, c), on(4, 1), on(4, 2), on(4, 3), on(4, 4)\}$

The intended Herbrand Interpretation is $\models_{HI} \{on(c, a), on(a, 1), on(b, 3)\}$
It defines the physical state of the system.

But generally, any subset of the Herbrand Base (i.e. selected atoms assumed to be true) defines a legal Herbrand Interpretation.

The Herbrand Theorems

Theorem 1 *Herbrand Theorem I: A set of clauses S (a formula in CNF with all variables universally quantified) is unsatisfiable iff it is unsatisfiable under any Herbrand Interpretation.*

Theorem 2 *Herbrand Theorem II: A set of clauses S (a formula in CNF with all variables universally quantified) is unsatisfiable iff there exists finite and unsatisfiable set S' of ground instances of clauses of S; $S' \subseteq S$.*

W word of rough explanation...

1. Herbrand interpretations restrict checking of unsatisfiability to finie/countable Herbrand Universes (no new symbols are introduced).
2. Using Herbrand Base is analogous to reducing the FOPC to PC.
3. Herbrand theorems are useful in proving satisfiability (in case of Dual Resolution) and unsatisfiability (in case of Resolution Theorem Proving) of sets of clauses in th S-Form.
4. Herbrand theorems are used in proofs of properties of Resolution Theorem Proving and Dual Resolution Theorem Proving.

Formulas Transformation Rules FOPC

Notation: $\Phi[X]$ — explicit occurrence of variable X in formula Φ.

Recall that $\forall$ is a generalized conjunction and $\exists$ is a generalized disjunction.

Basic rules for quantifiers (X does not occur in Ψ):

- $\forall X \Phi[X] \wedge \Psi \equiv \forall X (\Phi[X] \wedge \Psi)$,
- $\forall X \Phi[X] \vee \Psi \equiv \forall X (\Phi[X] \vee \Psi)$,
- $\exists X \Phi[X] \wedge \Psi \equiv \exists X (\Phi[X] \wedge \Psi)$,
- $\exists X \Phi[X] \vee \Psi \equiv \exists X (\Phi[X] \vee \Psi)$.

Generalized De Morgan Rules:

- $\neg(\forall X \Phi[X]) \equiv \exists X (\neg \Phi[X])$,
- $\neg(\exists X \Phi[X]) \equiv \forall X (\neg \Phi[X])$.

Distribution Rules for Quantifiers:

- $\forall X \Phi[X] \wedge \forall X \Psi[X] \equiv \forall X (\Phi[X] \wedge \Psi[X])$,
- $\exists X \Phi[X] \vee \exists X \Psi[X] \equiv \exists X (\Phi[X] \vee \Psi[X])$.

Auxiliary Rules with Renaming of Variables:

- $\forall X \Phi[X] \vee \forall X \Psi[X] \equiv \forall X \Phi[X] \vee \forall Y \Psi[Y] \equiv \forall X \forall Y (\Phi[X] \vee \Psi[Y])$,
- $\exists X \Phi[X] \wedge \exists X \Psi[X] \equiv \exists X \Phi[X] \wedge \exists Y \Psi[Y] \equiv \exists X \exists Y (\Phi[X] \wedge \Psi[Y])$.

Most important equivalent transformations — Analogs to Propositional Calculus

- $\neg\neg\phi \equiv \phi$ — double negation elimination,
- $\phi \land \psi \equiv \psi \land \phi$ — conjunction alternation,
- $\phi \lor \psi \equiv \psi \lor \phi$ — disjunction alternation,
- $(\phi \land \varphi) \land \psi \equiv \phi \land (\varphi \land \psi)$ — commutativity,
- $(\phi \lor \varphi) \lor \psi \equiv \phi \lor (\varphi \lor \psi)$ — commutativity,
- $(\phi \lor \varphi) \land \psi \equiv (\phi \land \psi) \lor (\varphi \land \psi)$ — distributive law,
- $(\phi \land \varphi) \lor \psi \equiv (\phi \lor \psi) \land (\varphi \lor \psi)$ — distributive law,
- $\phi \land \phi \equiv \phi$ — idempotency,
- $\phi \lor \phi \equiv \phi$ — idempotency,
- $\phi \land \bot \equiv \bot$, $\phi \land \top \equiv \phi$ — identity,
- $\phi \lor \bot \equiv \phi$, $\phi \lor \top \equiv \top$ — identity,
- $\phi \lor \neg\phi \equiv \top$ — *tertium non datur*; excluded middle,
- $\phi \land \neg\phi \equiv \bot$ — falsification,
- $\neg(\phi \land \psi) \equiv \neg(\phi) \lor \neg(\psi)$ — De Morgan rule,
- $\neg(\phi \lor \psi) \equiv \neg(\phi) \land \neg(\psi)$ — De Morgan rule,
- $\phi \Rightarrow \psi \equiv \neg\psi \Rightarrow \neg\phi$ — contraposition,
- $\phi \Rightarrow \psi \equiv \neg\phi \lor \psi$ — implication elimination.

Normal Forms

Definition 13 *Formula Φ is in Prenex Normal Form iff it is represented as*

$$(Q_1X_1)\ldots(Q_nX_n)(M),$$

where (Q_kX_k) (for $1 \leq k \leq n$) is $(\forall X_k)$ or $(\exists X_k)$, and M — the so called Matrix — is a quantifier-free formula.

Transforming to Prenex Normal Form (PNF):

1. Elimination of $\Leftrightarrow$ i $\Rightarrow$ by equivalent transformations,
2. Move all negation signs directly before predicate symbols (De Morgan Rules – also for quantifiers; double negation elimination),
3. Move all quantifiers into the prefix (prenex) using the distributivity rules.

Example:

Transform: $\exists Z \forall X((r(Z) \wedge p(X) \Rightarrow \exists Y q(X,Y))$ to PNF

$\exists Z \forall X((r(Z) \wedge p(X) \Rightarrow \exists Y q(X,Y)) \equiv$

$\exists Z \forall X(\neg(r(Z) \wedge p(X) \vee \exists Y q(X,Y)) \equiv$

$\exists Z \forall X((\neg r(Z) \vee \neg p(X) \vee \exists Y q(X,Y)) \equiv$

$\exists Z \forall X \exists Y(\neg r(Z) \vee \neg p(X) \vee q(X,Y))$

Skolem Normal Form and Skolemization

Definition 14 *Formula Φ is in Skolem Normal Form if it is in Prenex Normal Form and:*

- *There are no existential quantifiers in the prenex (prefix),*
- *the matrix M in in CNF.*

Transforming to Skolem Normal Form:

1. Transform the formula to PNF,
2. Transform the matrix M to CNF,
3. Sequentially transform the prenex $(Q_1X_1)\dots(Q_nX_n)$, until all existential quantifiers are eliminated:
   - if there is in the prenex $Q_rX_r = \exists X_r$ and there are no preceding universal quantifiers, then in the matrix M we replace X_R with an arbitrary new constant c, which does not appear in M and we delete $\exists X_r$ from the prenex,
   - if before $Q_rX_r = \exists X_r$ there occur s universal quantifiers $(\forall X_{j_1})\dots(\forall X_{j_s})$, where $(1 \leq j_1 \leq j_s \leq r)$, then in the matrix M we replace X_R with a new term of arity s, i.e. $f(X_{j_1},\dots,X_{j_s})$ (f does not appear in M) and we delete $\exists X_r$ from the prenex.

Example:

$\exists Z \forall X \exists Y(\neg r(Z) \vee \neg p(X) \vee q(X,Y)) \equiv \forall X(\neg r(c) \vee \neg p(X) \vee q(X,f(X)))$,
where c is a new constant and f is a functional symbol.

Finally, the so called S-Form (Clausal Form) is: $\{\neg r(c) \vee \neg p(X) \vee q(X,f(X))\}$
Note that all the quantifiers can be omitted, since now all the variables are universally quantified.

- Theorem Proving — Verification of Logical Consequence:

$$\Delta \models H;$$

- Method of Theorem Proving: Automated Inference —- Derivation:

$$\Delta \vdash H;$$

- SAT (checking for models) — satisfiability:

$$\models_I H \quad \text{(if such I exists)};$$

- un-SAT verification — unsatisfiability:

$$\not\models_I H \quad \text{(for any I)};$$

- Tautology verification (The Dual Resolution and completeness verification):

$$\models H$$

- Unsatisfiability verification (The Resolution Method)

$$\not\models H$$

Two principal issues:

- valid inference rules — checking:

$$(\Delta \vdash H) \quad \longrightarrow \quad (\Delta \models H)$$

- complete inference rules — checking:

$$(\Delta \models H) \quad \longrightarrow \quad (\Delta \vdash H)$$

The Deduction Theorems

Theorem 3 *Let $\Delta_1, \Delta_2, \ldots \Delta_n$ and Ω are logical formulas. Ω is their logical consequence iff $\Delta_1 \wedge \Delta_2 \wedge \ldots \Delta_n \Rightarrow \Omega$ is a tautology.*

Theorem 4 *Let $\Delta_1, \Delta_2, \ldots \Delta_n$ and Ω are logical formulas. Ω is their logical consequence iff $\Delta_1 \wedge \Delta_2 \wedge \ldots \Delta_n \wedge \neg\Omega$ is invalid (false under any interpretation).*

Theorem proving: having $\Delta_1, \Delta_2, \ldots \Delta_n$ assumed to be true show that so is Ω. Hence:

$$\Delta_1 \wedge \Delta_2 \wedge \ldots \Delta_n \models \Omega$$

Basic methods for theorem proving:

- evaluation of all possible interpretations (the 0-1 method),
- direct proof (forward chaining) – derivation of Ω from initial axioms; KRR: Rule-Based Systems, Expert Systems, Inference Graphs,...
- search for proof (backward chaining) – search for derivation of Ω from initial axioms; KRR: Backtracking Search, Abductive Reasoning, Diagnostic Systems, Question-Answering Systems, Prolog,...
- proving tautology – from the Deduction Theorem 1 we prove that $\Delta_1 \wedge \Delta_2 \wedge \ldots \Delta_n \Rightarrow \Omega$ is a tautology (Dual Resolution),
- indirect proof – through constraposition: $\neg\Omega \Rightarrow \neg(\Delta_1 \wedge \Delta_2 \wedge \ldots \Delta_n)$.
- Reductio ad Absurdum; basing on Deduction Theorem 2 we show that $\Delta_1 \wedge \Delta_2 \wedge \ldots \Delta_n \wedge \neg\Omega$. is unsatisfiable (Resolution Theorem Proving)

Some most important Inference Rules (Fitch)

- AND Introduction (AI):

$$\frac{\phi_1, \ldots, \phi_n}{\phi_1 \wedge \ldots \wedge \phi_n}$$

- AND Elimination (AE):

$$\frac{\phi_1 \wedge \ldots \wedge \phi_n}{\phi_i}$$

- OR Introduction (OI):

$$\frac{\phi_i}{\phi_1 \vee \ldots \vee \phi_n}$$

- OR Elimination (OE):

$$\frac{\phi_1 \vee \ldots \vee \phi_n, \phi_1 \Rightarrow \psi, \ldots \phi_n \Rightarrow \psi}{\psi}$$

- Negation Introduction (NI):

$$\frac{\phi \Rightarrow \psi, \phi \Rightarrow \neg\psi}{\neg\phi}$$

- Negation Elimination (NE):

$$\frac{\neg\neg\phi}{\phi}$$

- Implication Introduction (II):

$$\frac{\phi \vdash \psi}{\phi \Rightarrow \psi}$$

- Implication Elimination (IE):

$$\frac{\phi, \phi \Rightarrow \psi}{\psi}$$

- Equivalence Introduction (EI),
- Equivalence Elimination (EE)

Extra Rules for Quantifiers (Fitch)

Universal Introduction (UI)

$$\frac{\Phi}{\forall X : \Phi}$$

Universal Elimination (UE)

$$\frac{\forall X : \Phi[X]}{\Phi[t]}$$

gdzie $t \in \mathrm{TER}$.

Existential Introduction (EI)

$$\frac{\Phi[t]}{\exists X : \Phi[X]}$$

Existential Elimination (EE)

$$\frac{\exists X : \Phi[X], \quad \forall Y : (\Phi[Y] \Rightarrow \Psi)}{\Psi}$$

where the variable Y does not occur in formula Ψ.

Resolution Theorem Proving

1. Instead of proving that:

$$\{\Delta_1, \Delta_2, \ldots, \Delta_n\} \models H$$

   we prove unsatisfiability of:

$$\{\Delta_1, \Delta_2, \ldots, \Delta_n\} \cup \{\neg H\}$$

2. The initial formulas are transformed into equivalent Prenex Normal Form (PNF).
3. The matrix M of the formula is transformed to CNF.
4. By the skolemization procedure we eliminate all the existential quantifiers.
5. Since all the variables are universally quantified, the prenex contaning quantification can be removed.
6. As the result we obtain a set of clauses: the so-called S-Form.
7. Using the Resolution Metod we attempt to derive and empty clause (always false).

Resolution Rule for clauses $C_1 = \phi \vee q_1$ and $C_2 = \varphi \vee \neg q_2$; σ is a unifying substitution (mgu):

$$\frac{\phi \vee q_1, \varphi \vee \neg q_2}{\phi\sigma \vee \varphi\sigma}$$

We need also the Factorization Rule:

$$\frac{C}{C\theta}$$

The Factorization Rule is necessary for the cases such as:

$$\{p(X) \vee p(Y), \neg p(U) \vee \neg p(V)\}$$

Example: The Barber Paradox

There is a barber who was ordered to shave anyone who does not shave himself. Should he shave himself or not?

A simple problem formalization in FOPC:

A. $\forall X \neg \text{shaves}(X, X) \Rightarrow \text{shaves}(\text{barber}, X)$ — anyone who does not shave himself is shaved by the barber.

B. $\forall Y \text{shaves}(\text{barber}, Y) \Rightarrow \neg \text{shaves}(Y, Y)$ — anyone who is not shaved by the barber shaves himself.

Transformation to the S-Form:

- $C_1 = \text{shaves}(X, X) \vee \text{shaves}(\text{barber}, X)$,
- $C_2 = \neg \text{shaves}(\text{barber}, Y) \vee \neg \text{shaves}(Y, Y)$.

Niech $\theta = \{X/\text{barber}, Y/\text{barber}\}$.

$$C_1\theta = \text{shaves}(\text{barber}, \text{barber})$$

$$C_2\theta = \neg \text{shaves}(\text{barber}, \text{barber})$$

W wyniku rezolucji mamy:

$$\frac{\text{shaves}(\text{barber}, \text{barber}), \neg \text{shaves}(\text{barber}, \text{barber})}{\bot}$$

What is the conclusion then?

What does this solution consist in?

Some Concluding Remarks on Resolution Method in FOPC

Since the resolvent is a logical consequence of parent clauses, we have $C_1, C_2 \models \perp$, i.e. the initial statement is in fact internally inconsistent.

To conclude, the Resolution Rule, augmented with Factorization Rule, constitute a tool for theorem proving – the Resolution Theorem Proving Method which is:

- based on refutation — an empty clause (always false) is to be derived from assumptions completed with negated conclusion,
- sound — any conclusion derived with resolution (and factorization) is sound,
- complete — in the sense that an empty clause can always be deduced from an unsatisfiable set of clauses.

Resolution theorem proving is based on using the clausal form, i.e. quantifier-free First-Order Logic CNF formula. Hence it is especially convenient for systems which are or can be easily transformed into CNF.

In Knowledge-Based Systems resolution is applied for: applications:

- proving satisfaction of preconditions of rules in order to check if a selected rule can be fired,
- proving attainability of goals – The Question-Answering Systems,
- checking for inconsistent rules.

Resolution is also the basic rule implemented in all PROLOG systems
A PROLOG program is a set of Horn clauses.

Dual Resolution Method

1. We prove validity rather than unsatisfiability,
2. The formula is transformed into the Prenex Normal Form.
3. The matrix is transformed into DNF rather than CNF!
4. Next – dual skolemization – is applied; all universal quantifiers are eliminated; the procedure follows.

Let $Q_1X_1\ Q_2X_2\ \ldots\ Q_nX_n\Psi$ be the prenex normal form obtained from the initial formula, where Q_i, $i = 1, 2, \ldots, n$ are all the quantifiers and Ψ is the quantifier-free matrix of the formula in DNF. Assume that Q_i is the first universal quantifier encountered when scanning the prefix of the formula from left to the right. Now there are two possibilities:

1) If no existential quantifier occurs before Q_i then all the occurrences of variable X_i in Ψ are replaced with a new constant c (c cannot occur in Ψ) and Q_iX_i is removed from the prefix.

2) If $Q_{k_1}, Q_{k_2}, \ldots, Q_{k_j}$ are all the existential quantifiers occurring before Q_i, then all the occurrences of X_i in Ψ are replaced with a term of the form $f(X_{k_1}, X_{k_2}, \ldots, X_{k_j})$, where f is a new function symbol, and Q_iX_i is removed from the prefix.

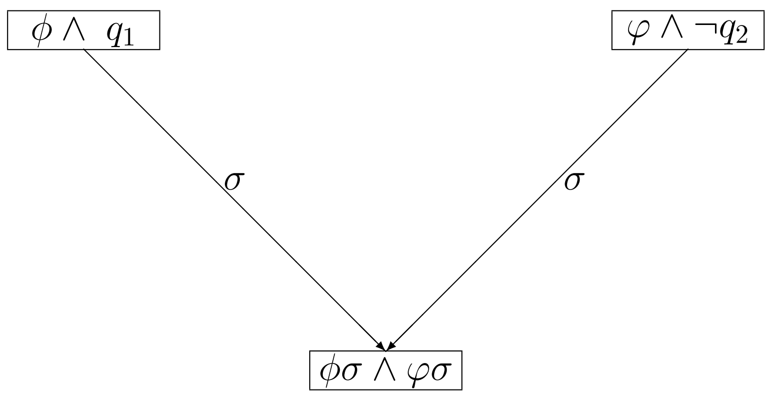
In this way all the universal quantifiers are eliminated from the prefix, and, since all the variables are existentially quantified, the prefix can be omitted.

Theorem 5 *Let Ω be a logical formula and let Φ be its quantifier-free minterm form. Ω is a tautology if and only if Φ is a tautology.*

Dual Resolution Rule

Let there be given two minterms, $M_1 = \phi \wedge q_1$ and $M_2 = \varphi \wedge \neg q_2$. It is important that q_1 and $\neg q_2$ are either complementary literals or there exists a most general unifier σ, such that $q_1\sigma$ and $q_2\sigma$ are identical, an so $q_1\sigma$ and $\neg q_2\sigma$ are complementary. The bd-resolution rule (or bd-resolution principle) allows to generate a new simple formula $M = \phi\sigma \wedge \varphi\sigma$; the complementary literals are removed.

A graphical presentation is given below:



What is important and constitutes a principal difference with respect to classical resolution rule is that the disjunction of the parent minterms is a logical consequence of the generated result; this is schematically presented below.

$$\boxed{\phi\sigma \wedge \varphi\sigma} \models \boxed{\phi \wedge \ q_1} \vee \boxed{\varphi \wedge \neg q_2}$$

Definition 15 Dual Resolution Rule. *Let $M_1 = \phi \wedge q_1$ and $M_2 = \varphi \wedge \neg q_2$ are two arbitrary minterms. Let σ be a mgu for q_1 and q_2. The Backward Dual Resolution Rule is an inference rule of the form:*

$$\frac{\phi \wedge q_1; \varphi \wedge \neg q_2}{\phi\sigma \wedge \varphi\sigma}. \tag{1}$$

Obviously, the produced formula is not a logical consequence of the parent formulae. The rule works in a certain sense backwards – the disjunction of the parent minterms is a logical consequence of the result, i.e. there is $M \models M_1 \vee M_2$.

By analogy to classical resolution theorem proving, in theorem proving with bd-resolution factorization is also a necessary additional rule to assure completeness.

Let M be any minterm such that two or more literals of M can be unified with some most general unifier θ; in this case M is a logical consequence of $M\theta$ ($M\theta \models M$) and $M\theta$ is called a *factor* of M. The rule

$$\frac{M}{M\theta}$$

is called *factorization*. Factorization is a complementary, but necessary rule to assure completeness of bd-resolution theorem proving.

BD-Derivation

Now, let us define the way in which one can generate a sequence of bd-resolvents starting from some initial normal formula

$$\Psi = \psi_1 \vee \psi_2 \vee \ldots \vee \psi_m \tag{2}$$

i.e. the so-called *bd-derivation*. This is done in the following way.

Definition 16 *A* bd-derivation *(or derivation, for short) of a simple formula ψ from a normal formula Ψ given by (2) is any sequence of simple formulae $\psi^1, \psi^2, \ldots, \psi^k$, such that:*

- *for any $j \in \{1, 2, \ldots, k\}$ ψ^j is either a factor of some ψ^i or a bd-resolvent of simple formulae ψ^i, $\psi^{i'}$, where either $i \leq j$ or $\psi^i \in \Psi$ and $i' \leq j$ or $\psi^{i'} \in \Psi$,*
- $\psi = \psi^k$.

Formula ψ is said to be bd-derived *from Ψ.*

A formula ψ can be derived from some normal formula Ψ by generating a sequence of simple formulae, such that any formula in the sequence is either a factor of, or a bd-resolvent of some earlier generated formulae (or the ones in Ψ); any formula in the sequence is said to be bd-derived from Ψ, and if ψ appears as the last formula in the above sequence, then it is bd-derived from Ψ as well. This will be denoted shortly as $\Psi \vdash_{BDR} \psi$. For simplicity, in case of no ambiguity, we shall also say that ψ is derived from Ψ and we shall write $\Psi \vdash \psi$.

Exmaple Application of Dual Resolution

Consider the foollwoing set of rules:

$$
\begin{array}{rcl}
\texttt{UNIV\_MEMBER}(X) \wedge \texttt{ENROLLED}(X) \wedge \texttt{HAS\_BS\_DEGREE}(X) & \longrightarrow & \texttt{STATUS}(X, GraduateStudent) \\
\texttt{UNIV\_MEMBER}(X) \wedge \texttt{ENROLLED}(X) \wedge \neg\texttt{HAS\_BS\_DEGREE}(X) & \longrightarrow & \texttt{STATUS}(X, Undergraduate) \\
\texttt{UNIV\_MEMBER}(X) \wedge \neg\texttt{ENROLLED}(X) \wedge \texttt{HAS\_BS\_DEGREE}(X) & \longrightarrow & \texttt{STATUS}(X, Staff) \\
\neg\texttt{ENROLLED}(X) \wedge \neg\texttt{HAS\_BS\_DEGREE}(X) & \longrightarrow & \texttt{STATUS}(X, NonAcademic) \\
\neg\texttt{UNIV\_MEMBER}(X) & \longrightarrow & \texttt{STATUS}(X, NonAcademic)
\end{array}
$$

Using the bd-resolution one can produce most general formulae specifying logical completeness of preconditions of the rules, i.e. showing that in *any* case of input data at least one rule can be fired (as it covers the case).

From preconditions of the first and second rules, we obtain a bd-resolvent:

$$\psi_1 = \texttt{UNIV\_MEMBER}(X) \wedge \texttt{ENROLLED}(X).$$

From preconditions of first and third rule we have a bd-resolvent of the form:

$$\psi_2 = \texttt{UNIV\_MEMBER}(X) \wedge \texttt{HAS\_BS\_DEGREE}(X).$$

Note that $\psi_1 \vee \psi_2$ specifies the *positive* cases covered by the system (academic persons). The system is *specifically logically complete* with respect to $\psi = \psi_1 \vee \psi_2$.

Now, apply the non-academic cases specification given by rules four and five. For intuition, the above rules cover any non UNIV_MEMBER nor anyone not ENROLLED and such that HAS_BS_DEGREE is not satisfied.

By applying bd-resolution to ψ_1 and precondition of the fourth rule one obtains $\texttt{UNIV\_MEMBER}(X) \wedge \neg\texttt{HAS\_BS\_DEGREE}(X)$, and by further bd-resolving with ψ_2 one obtains $\texttt{UNIV\_MEMBER}(X)$. Finally, after resolving the result with the preconditions of the fifth rule one obtains the empty formula $\top$ (always true). Hence, the disjunction of the preconditions of the above rules is tautology.

Soundness and Completeness of BD-Resolution

The following theorem assures soundness of bd-resolution.

Theorem 6 **Soundness of bd-resolution rule**. *Let Ψ be a formula of the form (2) and let $\psi_i = \phi \wedge q_1$ and $\psi_j = \varphi \wedge \neg q_2$ be any two minterms of Ψ defined by (1). Moreover, let $\psi = \phi\sigma \wedge \varphi\sigma$ be the bd-resolvent of them. Then*

$$\psi \models \psi_i \vee \psi_j \tag{3}$$

and $\psi \models \Psi$.

Theorem 7 **Soundness of bd-derivation**. *Let Ψ be a formula of the form (2) and let ψ be a simple formula obtained by bd-derivation from Ψ. Then $\psi \models \Psi$.*

The theorem assuring completeness of bd-resolution can be stated as follows.

Theorem 8 ***Completeness Theorem*** *Let Ψ be any normal formula, and let ϕ be some simple formula. Assume that Ψ and ϕ have no variables in common, i.e. $FV(\Psi) \cap FV(\phi) = \emptyset$[1]. If*

$$\phi \models \Psi, \tag{4}$$

then there exists a bd-derivation of a simple formula ψ from Ψ, such that

$$\phi \models \psi. \tag{5}$$

Moreover, if ϕ is satisfiable, then there exists a substitution θ such that

$$[\psi\theta] \subseteq [\phi], \tag{6}$$

i.e. the derived formula ψ subsumes ϕ.

[1]If no, a simple renaming of variables may be necessary.