



# Logic for Computer Science. Knowledge Representation and Reasoning.

Lecture Notes  
for  
Computer Science Students  
Faculty EAIIB-IEiT AGH



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## Inference and Theorem Proving in Propositional Calculus

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- Tasks and Models of Automated Inference,
- Theorem Proving models,
- Some important Inference Rules,
- Theorems of Deduction: 1 and 2,
- Models of Theorem Proving,
- Examples of Proofs,
- The **Resolution Method**,
- The **Dual Resolution Method**,
- Logical Derivation,
- The Semantic Tableau Method,
- Constructive Theorem Proving: The Fitch System,
- Example: The Unicorn,
- Looking for Models: Towards SAT.

## Logic for KRR — Tasks and Tools

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- Theorem Proving — Verification of Logical Consequence:

$$\Delta \models H;$$

- Method of Theorem Proving: Automated Inference — Derivation:

$$\Delta \vdash H;$$

- SAT (checking for models) — satisfiability:

$$\models_I H \quad (\text{if such } I \text{ exists});$$

- un-SAT verification — unsatisfiability:

$$\not\models_I H \quad (\text{for any } I);$$

- Tautology verification (completeness):

$$\models H$$

- Unsatisfiability verification

$$\not\models H$$


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Two principal issues:

- valid inference rules — checking:

$$(\Delta \vdash H) \longrightarrow (\Delta \models H)$$

- complete inference rules — checking:

$$(\Delta \models H) \longrightarrow (\Delta \vdash H)$$


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## Two Possible Fundamental Approaches: Checking of Interpretations versus Logical Inference

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Two basic approaches – reasoning paradigms:

- **systematic evaluation of possible interpretations** — the 0-1 method; problem — **combinatorial explosion**; for  $n$  propositional variables we have  $2^n$  interpretations!
- **logical inference** — **derivation** — with rules preserving **logical consequence**.

Notation: formula  $H$  (a Hypothesis) is derivable from  $\Delta$  (a Knowledge Base; a set of domain axioms):

$$\Delta \vdash H$$

This means that there exists a **sequence of applications** of inference rules, such that  $H$  is *mechanically* derived from  $\Delta$ .

Two principal issues in logical knowledge-based systems:

$$\Delta \vdash H \quad \text{versus} \quad \Delta \models H$$

i.e.

- is the derived formula **valid**?
- can any valid formula be **derived**?

## An example derivation - for intuition

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Just for intuition, let us consider an example of **constructive proof** by **linear derivation**:

$$\phi = (p \Rightarrow q) \wedge (r \Rightarrow s),$$

$$\varphi = (p \wedge r) \Rightarrow (q \wedge s).$$

This time we perform **derivation** of  $\varphi$  from  $\phi$ :

$$\phi \vdash \varphi$$

A rough outline of derivation steps:

- |     |                                         |                                            |
|-----|-----------------------------------------|--------------------------------------------|
| 1.  | $p \Rightarrow q$                       | by assumption;                             |
| 2.  | $r \Rightarrow s$                       | by assumption;                             |
| 3.  | $\underline{p \wedge r}$                | we introduce an assumption;                |
| 4.  | $p$                                     | elimination of conjunction from (3);       |
| 5.  | $q$                                     | Modus Ponens (1) and (4);                  |
| 6.  | $r$                                     | elimination of conjunction from (3; )      |
| 7.  | $s$                                     | Modus Ponens (2) and (6);                  |
| 8.  | $q \wedge s$                            | conjunction introduction from (5) and (7); |
| 9.  | $(p \wedge r) \vdash (q \wedge s)$      | the derivation based on assumption (3);    |
| 10. | $(p \wedge r) \Rightarrow (q \wedge s)$ | implication introduction based on (9)      |

Obviously, there is also:

$$\phi \models \varphi$$

**But why?**

## Some more important inference rules **!?! Tips tricks !?!**

- $\frac{\alpha}{\alpha \vee \beta}$  — Disjunction Introduction,
- $\frac{\alpha, \beta}{\alpha \wedge \beta}$  — Conjunction Introduction,
- $\frac{\alpha \wedge \beta}{\alpha}$  — Conjunction Elimination,
- $\frac{\alpha, \alpha \Rightarrow \beta}{\beta}$  — Modus Ponens (modus ponendo ponens); implication elimination (EI),
- $\frac{\alpha \Rightarrow \beta, \neg \beta}{\neg \alpha}$  — Modus Tollens (modus tollendo tollens),
- $\frac{\alpha \vee \beta, \neg \alpha}{\beta}$  — Modus Tollendo Ponens,
- $\frac{\alpha \oplus \beta, \alpha}{\neg \beta}$  — Modus Ponendo Tollens,
- $\frac{\alpha \Rightarrow \beta, \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma}$  — Transitivity Rule,
- $\frac{\alpha \vee \gamma, \neg \gamma \vee \beta}{\alpha \vee \beta}$  — **Resolution Rule**,
- $\frac{\alpha \wedge \gamma; \neg \gamma \wedge \beta}{\alpha \wedge \beta}$  — **Dual Resolution Rule**; (backward) dual resolution (works backwards); also termed *consolution*,
- $\frac{\alpha \Rightarrow \beta, \gamma \Rightarrow \delta}{(\alpha \vee \gamma) \Rightarrow (\beta \vee \delta)}$  — Constructive Dilemma I,
- $\frac{\alpha \Rightarrow \beta, \gamma \Rightarrow \delta}{(\alpha \wedge \gamma) \Rightarrow (\beta \wedge \delta)}$  — Constructive Dilemma II.

## The Deduction Theorems

**Theorem 1** Let  $\Delta_1, \Delta_2, \dots, \Delta_n$  and  $\Omega$  are logical formulas.  $\Omega$  is their logical consequence *iff*  $\Delta_1 \wedge \Delta_2 \wedge \dots \wedge \Delta_n \Rightarrow \Omega$  is a tautology.

**Theorem 2** Let  $\Delta_1, \Delta_2, \dots, \Delta_n$  and  $\Omega$  are logical formulas.  $\Omega$  is their logical consequence *iff*  $\Delta_1 \wedge \Delta_2 \wedge \dots \wedge \Delta_n \wedge \neg\Omega$  is invalid (false under any interpretation).

Theorem proving: having  $\Delta_1, \Delta_2, \dots, \Delta_n$  assumed to be true show that so is  $\Omega$ . Hence:

$$\Delta_1 \wedge \Delta_2 \wedge \dots \wedge \Delta_n \models \Omega$$

Basic methods for theorem proving:

- evaluation of all possible interpretations (the 0-1 method),
- **direct proof** (forward chaining) – derivation of  $\Omega$  from initial axioms;  
KRR: **Rule-Based Systems, Expert Systems, Inference Graphs,...**
- **search for proof** (backward chaining) – search for derivation of  $\Omega$  from initial axioms; KRR: **Backtracking Search, Abductive Reasoning, Diagnostic Systems, Question-Answering Systems, Prolog,...**
- **proving tautology** – from the Deduction Theorem 1 we prove that  $\Delta_1 \wedge \Delta_2 \wedge \dots \wedge \Delta_n \Rightarrow \Omega$  is a tautology,
- **indirect proof** – through constraposition:  
 $\neg\Omega \Rightarrow \neg(\Delta_1 \wedge \Delta_2 \wedge \dots \wedge \Delta_n)$ .
- **Reductio ad Absurdum**; basing on Deduction Theorem 2 we show that  $\Delta_1 \wedge \Delta_2 \wedge \dots \wedge \Delta_n \wedge \neg\Omega$  is unsatisfiable

## Examples

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**Direct proof:**  $(p \Rightarrow r) \wedge (q \Rightarrow s) \wedge (\neg r \vee \neg s) \models (\neg p \vee \neg q)$ :

1.  $p \Rightarrow r$       assumption,
2.  $q \Rightarrow s$       assumption,
3.  $\neg r \vee \neg s$     assumption,
4.  $s \Rightarrow \neg r$     implication reconstruction; through equivalence to 3,
5.  $q \Rightarrow \neg r$     transitivity 2 and 4,
6.  $\neg p \vee r$       EI from 1,
7.  $\neg q \vee \neg r$     EI from 5
8.  $\neg p \vee \neg q$     by resolution rule from 6 and 7.

**Proving tautology:**  $[p \Rightarrow (q \Rightarrow r)] \models [q \Rightarrow (p \Rightarrow r)]$ .

We transform the formula  $[p \Rightarrow (q \Rightarrow r)] \Rightarrow [q \Rightarrow (p \Rightarrow r)]$  and through elimination of implications we obtain  $\alpha \vee \neg\alpha$ .

**Indirect proof:**  $p \models \neg q \Rightarrow \neg(p \Rightarrow q)$

1.  $\neg(\neg q \Rightarrow \neg(p \Rightarrow q))$       assumption (contraposition),
2.  $\neg(q \vee \neg(p \Rightarrow q))$       EI,
3.  $(\neg q \wedge (p \Rightarrow q))$       De Morgan rule,
4.  $\neg q$       CE,
5.  $p \Rightarrow q$       CE from 3,
6.  $\neg p \vee q$       EI from 5,



7.  $q \vee \neg p$                       commutativity from 6,  
8.  $\neg p$                               RR from 4 and 7.

**Reductio ad Absurdum:**  $(p \vee q) \wedge \neg p \models q$

1.  $p \vee q$       assumption,
2.  $\neg p$         assumption,
3.  $\neg q$         assumption (negation of the hypothesis),
4.  $q$          RR to 1 and 2
5.  $\perp$         from 3 and 4.

## Example: Logical Consequence – EX-LCV16

$$\frac{(p \Rightarrow q) \wedge (r \Rightarrow s)}{(p \vee r) \Rightarrow (q \vee s)}$$

Let us put:

$$\phi = (p \Rightarrow q) \wedge (r \Rightarrow s)$$

and

$$\varphi = (p \vee r) \Rightarrow (q \vee s),$$

So we have to check if:

$$\phi \models \varphi. \tag{1}$$

$p$	$q$	$r$	$s$	$p \Rightarrow q$	$r \Rightarrow s$	$(p \Rightarrow q) \wedge (r \Rightarrow s)$	$p \vee r$	$q \vee s$	$(p \vee r) \Rightarrow (q \vee s)$
0	0	0	0	1	1	1	0	0	1
0	0	0	1	1	1	1	0	1	1
0	0	1	0	1	0	0	1	0	0
0	0	1	1	1	1	1	1	1	1
0	1	0	0	1	1	1	0	1	1
0	1	0	1	1	1	1	0	1	1
0	1	1	0	1	0	0	1	1	1
0	1	1	1	1	1	1	1	1	1
1	0	0	0	0	1	0	1	0	0
1	0	0	1	0	1	0	1	1	1
1	0	1	0	0	0	0	1	0	0
1	0	1	1	0	1	0	1	1	1
1	1	0	0	1	1	1	1	1	1
1	1	0	1	1	1	1	1	1	1
1	1	1	0	1	0	0	1	1	1
1	1	1	1	1	1	1	1	1	1

From columns 7 and 10 we conclude that **there is logical consequence** (but no equivalence —see rows 7, 10, 12 i 15).

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## The Resolution Method

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1. Problem:

$$\Delta \models H$$

2. From Deduction Theorem 2:

$$\Delta \cup \neg H$$

should be unsatisfiable.

3. Transform  $\Delta \cup \neg H$  to CNF.

4. Using the RR derive an empty formula  $\perp$ .

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### Example:

1. Problem:

$$(p \Rightarrow q) \wedge (r \Rightarrow s) \models (p \vee r) \Rightarrow (q \vee s)$$

2. From Deduction Theorem 2 — show that:

$$[(p \Rightarrow q) \wedge (r \Rightarrow s)] \cup \neg[(p \vee r) \Rightarrow (q \vee s)]$$

is unsatisfiable.

3. After transformation to CNF we have:

$$\{\neg p \vee q, \neg r \vee s, p \vee r, \neg q, \neg s\}$$

4. Derive  $\perp$ .

## Dual Resolution Method

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1. Problem:

$$\Delta \models H$$

2. From Deduction Theorem 1 show that:

$$\Delta \Rightarrow H$$

is a tautology.

3. Transform  $\Delta \Rightarrow H$  to DNF.

4. Using the DRR derive an empty formula — the always true one  $\top$ .

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### Example:

1. Problem:

$$(p \Rightarrow q) \wedge (r \Rightarrow s) \models (p \vee r) \Rightarrow (q \vee s)$$

2. From Deduction Theorem 1 show that:

$$[(p \Rightarrow q) \wedge (r \Rightarrow s)] \Rightarrow [(p \vee r) \Rightarrow (q \vee s)]$$

is a tautology.

3. After transformation to DNF we have:

$$\{p \wedge \neg q; r \wedge \neg s; \neg p \wedge \neg r; q; s\}$$

4. Using the DRR derive an empty formula — the always true one  $\top$ .

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## Example of Resolution Derivation

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**A** – signal from process,

**P** – signal added to a queue,

**B** – signal blocked by process,

**D** – signal received by process,

**S** – state of the process saved,

**M** – signal mask read,

**H** – signal management procedure activated,

**N** – procedure executed in normal mode,

**R** – process restart from context,

**I** – process must re-create context.

Rules — axiomatization:

$A \longrightarrow P,$

$P \wedge \neg B \longrightarrow D,$

$D \longrightarrow S \wedge M \wedge H,$

$H \wedge N \longrightarrow R,$

$H \wedge \neg R \longrightarrow I,$

Facts:

$A, \neg B, \neg R.$

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Application of RR to CNF:

$\{\neg A \vee P, \neg P \vee B \vee D, \neg D \vee S, \neg D \vee M, \neg D \vee H, \neg H \vee \neg N \vee R, \neg H \vee R \vee I, A, \neg B, \neg R\}$

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## Conclusions

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$P, D, S, M, H, I, \neg N.$

## Inference step; derivation

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Step of inference: single application of RR.

**Example:**

**Application of RR:**

$$\frac{\phi \vee \neg p, p \vee \psi}{\phi \vee \psi}$$

Notation:  $\{\phi \vee \neg p, p \vee \psi\} \vdash_R \phi \vee \psi$

**Definition 1 Derivation** A *derivation* of  $\phi$  from  $\Delta$  we call a sequence:

$$\phi_1, \phi_2 \dots \phi_k$$

such that:

- formula  $\phi_1$  is derivable from  $\Delta$  (in a single step):

$$\Delta \vdash \phi_1,$$

- every next formula is derivable from  $\Delta$  and the earlier-derived formulas:

$$\{\Delta, \phi_1, \phi_2, \dots, \phi_i\} \vdash \phi_{i+1}$$

for  $i = 2, 3, \dots, k - 1$ ,

- $\phi$  is the last formula:

$$\phi = \phi_k$$

Notation:  $\Delta \vdash \phi$ , and  $\phi$  is called *derivable from  $\Delta$* .



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## Set of Logical Consequences $C_n$

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**Definition 2** Let  $\Delta$  be set of formulas. The set of logical consequences is:

$$C_n(\Delta) = \{\phi : \Delta \models \phi\}$$

where every  $\phi$  contains (only) propositional symbols of  $\Delta$ .

**Lemma 1 Properties of  $C_n$**  There are:

- $\Delta \subseteq C_n(\Delta)$ ,
- *monotonicity* — if  $\Delta_1 \subseteq \Delta_2$ , then:

$$C_n(\Delta_1) \subseteq C_n(\Delta_2)$$

- $C_n(C_n(\Delta)) = C_n(\Delta)$  (the so-called fixed point).

Is the Fixed Point unique? Is it finitely defined? Is it finite?

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**Example:** Consider the following set of formulas:

$$\Delta = \{\neg(\neg p \wedge \neg r), r \Rightarrow q, \neg q, p \Rightarrow t, \neg(t \wedge \neg s)\}$$

Show that:

$$\Delta \models s$$

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## The Semantic Tableau Method

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Recall the notions of: an atom, a literal, a positive literal, a negative literal  $\{p, \neg p\}$ .

Recall that a formula  $p \wedge \neg p$  is always false. Formula  $p \vee \neg p$  is always true.

### Assumptions:

- we consider **satisfiability of a formula**,
- the starting point is the **formula in original form!** (it is not necessary to transform it into the CNF/DNF),
- by analysis and decomposition we search for a model; no model means **unsatisfiability**,
- we develop a tree (or a table):
  - for **conjunctive formals** we develop a single branch (a linear form),
  - for **disjunctive formulas** we develop branches,
- existence of a pair of complementary literals closes a given branch (falsifies),
- lack of complementary literals — leads to a model (satisfiability),
- closing each branch means **unsatisfiability** of the original formula.

### Example 1:

$$p \wedge (\neg q \vee \neg p)$$

### Example 2:

$$(p \vee q) \wedge (\neg p \wedge \neg q)$$

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## Examples

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### Example 1:

$$p \wedge (\neg q \vee \neg p)$$

$$p, \neg q \vee \neg p$$

$$p, \neg q \quad p, \neg p$$

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### Example 2:

$$(p \vee q) \wedge (\neg p \wedge \neg q)$$

$$p \vee q, \neg p \wedge \neg q$$

$$p \vee q, \neg p, \neg q$$

$$p, \neg p, \neg q \quad q, \neg p, \neg q$$

## Semantic Tableau Algorithm

Rules of transformation for conjunctive formulas (type  $\alpha$ ):

$\alpha$	$\alpha_1$	$\alpha_2$
$\neg\neg A$	$A$	
$A_1 \wedge A_2$	$A_1$	$A_2$
$\neg(A_1 \vee A_2)$	$\neg A_1$	$\neg A_2$
$\neg(A_1 \Rightarrow A_2)$	$A_1$	$\neg A_2$
$A_1 \Leftrightarrow A_2$	$A_1 \Rightarrow A_2$	$A_2 \Rightarrow A_1$

Rules of transformation for disjunctive formulas (type  $\beta$ ):

$\beta$	$\beta_1$	$\beta_2$
$B_1 \vee B_2$	$B_1$	$B_2$
$\neg(B_1 \wedge B_2)$	$\neg B_1$	$\neg B_2$
$B_1 \Rightarrow B_2$	$\neg B_1$	$B_2$
$\neg(B_1 \Leftrightarrow B_2)$	$\neg(B_1 \Rightarrow B_2)$	$\neg(B_2 \Rightarrow B_1)$

An Algorithm for developing the Semantic Tree:

- The Root: the initial formula (in original form; WFF),
- U (for leaves) contains literals only:
  - $p, \neg p \in U$  — stop/falsification; *else*
  - stop/a model found,
- For a conjunctive formula  $\alpha \in U$ :

$$U' = (U - \{\alpha\}) \cup \{\alpha_1, \alpha_2\}$$

- For a disjunctive formula  $\beta \in U$  we have **branching**:

$$U' = (U - \{\beta\}) \cup \{\beta_1\}$$

$$U'' = (U - \{\beta\}) \cup \{\beta_2\}$$

### Example:

#### 1. Problem:

$$(p \Rightarrow q) \wedge (r \Rightarrow s) \models (p \vee r) \Rightarrow (q \vee s)$$

#### 2. Based on the Deduction Theorem (2), it should be shown that:

$$[(p \Rightarrow q) \wedge (r \Rightarrow s)] \cup \neg[(p \vee r) \Rightarrow (q \vee s)]$$

is unsatisfiable.

#### 3. Transform to CNF. We have:

$$\{\neg p \vee q, \neg r \vee s, p \vee r, \neg q, \neg s\}$$

#### 4. Using *Resolution Rule* derive an empty clause — always false.

**Problem:** show that the following set of formulas is unsatisfiable with use of Semantic Tableau method.

$$[(p \Rightarrow q) \wedge (r \Rightarrow s)] \cup \neg[(p \vee r) \Rightarrow (q \vee s)]$$

In fact, we have a formula:

$$[(p \Rightarrow q) \wedge (r \Rightarrow s)] \wedge \neg[(p \vee r) \Rightarrow (q \vee s)]$$

## Constructive Theorem Proving: The Fitch System

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- AND Introduction (AI):

$$\frac{\phi_1, \dots, \phi_n}{\phi_1 \wedge \dots \wedge \phi_n}$$

- AND Elimination (AE):

$$\frac{\phi_1 \wedge \dots \wedge \phi_n}{\phi_i}$$

- OR Introduction (OI):

$$\frac{\phi_i}{\phi_1 \vee \dots \vee \phi_n}$$

- OR Elimination (OE):

$$\frac{\phi_1 \vee \dots \vee \phi_n, \phi_1 \Rightarrow \psi, \dots, \phi_n \Rightarrow \psi}{\psi}$$

- Negation Introduction (NI):

$$\frac{\phi \Rightarrow \psi, \phi \Rightarrow \neg\psi}{\neg\phi}$$

- Negation Elimination (NE):

$$\frac{\neg\neg\phi}{\phi}$$

- Implication Introduction (II):

$$\frac{\phi \vdash \psi}{\phi \Rightarrow \psi}$$

- Implication Elimination (IE):

$$\frac{\phi, \phi \Rightarrow \psi}{\psi}$$

- Equivalence Introduction (EI),

- Equivalence Elimination (EE)

## Example: Unicorn

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Given the following Knowledge Base (KB):

- If the unicorn is mythical, then it is immortal
- If the unicorn is not mythical, then it is a mortal mammal
- If the unicorn is either immortal or a mammal, then it is horned
- The unicorn is magical if it is horned

answer the following questions:

- Is the unicorn mythical? ( $M$ )
- Is it magical? ( $G$ )
- Is it horned? ( $H$ )

In terms of logic:

$$\text{KB} \models G, H, M$$

$$\text{KB} \vdash G, H, M$$

## Unicorn - Logical Model

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Definition of propositional variables:

- M: The unicorn is mythical
- I: The unicorn is immortal
- L: The unicorn is mammal
- H: The unicorn is horned
- G: The unicorn is magical

Building a **Logical Model** for the puzzle:

- If the unicorn is mythical, then it is immortal:

$$M \longrightarrow I$$

- If the unicorn is not mythical, then it is a mortal mammal:

$$\neg M \longrightarrow (\neg I \wedge L)$$

- If the unicorn is either immortal or a mammal, then it is horned:

$$(I \vee L) \longrightarrow H$$

- The unicorn is magical if it is horned:

$$H \longrightarrow G$$

Resulting Boolean formula (the **Knowledge Base**):

$$(M \longrightarrow I) \wedge (\neg M \longrightarrow (\neg I \wedge L)) \wedge ((I \vee L) \longrightarrow H) \wedge (H \longrightarrow G)$$



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## A Solution: Formal Derivation of Logical Consequences

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1.  $(M \longrightarrow I) \equiv (\neg M \vee I)$
2.  $(\neg M \longrightarrow (\neg I \wedge L)) \equiv (M \vee (\neg I \wedge L))$
3.  $(M \vee (\neg I \wedge L)) \equiv ((M \vee \neg I) \wedge (M \vee L))$
4.  $\neg M \vee I, M \vee L$
5.  $I \vee L$
6.  $I \vee L, (I \vee L) \longrightarrow H$
7.  $H$
8.  $H, H \longrightarrow G$
9.  $G$

So we have:

$$\text{KB} \vdash H \wedge G$$

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### Questions:

- What about M (mythical), I (immortal) and L (mammal)?
  - What are the exact models? What combinations are admissible?
  - How many models do we have?
  - What is the CNF of the original formula?
  - What is the DNF of the original formula?
  - Resolution, Dual Resolution, Semantic Tableau, Fitch System,...
- Try each one; which one you prefer?