Twierdzenie Goedla i inne rozmaitości

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Rysunek: Lecture: Logic I

Axioms: Additivity:

1. $a + 0 = a$
2. $a + S(b) = S(a + b)$

Multiplication:

1. $a \bullet 0 = 0$
2. $a \bullet S(b) = a \bullet b + a$

Further properties of successor-relation

1. $S(x) \neq 0$
2. $S(x) = S(y) \rightarrow x = y$
3. $x \neq 0 \rightarrow \exists y(x = S(y))$

$\mathcal{M} = \langle \{1, 2, 3 \ldots\}, \leq \rangle$ as a standard model, but there are also other ones!

Godel's theorem

Theorem

Diagonalization lemma *For each formula ϕ of PA there is such a new predicate of a language of PA Prov such that it holds:*

$$PA \vdash \phi \iff \neg Prov(G(\phi)), \tag{1}$$

where G is a Goedel's number of ϕ.

Theorem

I incompleteness Goedel's theorem

Assume that PA is ω-consistent theory and:

1. $PA \vdash p \iff \neg Prov(G(p))$,
2. $Prov(p) \to p$,

than neither:

1. *non* $PA \vdash p$ ***nor*** *non* $PA \vdash \neg p$.

II Incompleteness theorem

Theorem

II Incompleteness theorem *If T is a consistent formal theory which is able to formalize a certain part of arithmetic, then T does not prove its own consistency.*

Undecidable theories

Definition

A theory is **decidable** if a set of its theorems is recursive. (If we do not have such Goedel's sentences in this theory.)

1. PA arithmetic (1930, Goedel)
2. predicate Calculus (1936, Church)
3. lattice theory (Tarski, 1949)
4. **ZF** set theory (Tarski)
5. theory of the structure $\langle Q, +, \bullet \rangle$ (J. Robinson)
6. Robinson's arithmetic ....

Pewna ilustracja z życia

Pewna rzeczy dadzą się udowodnić:

1. Nie jestem w stanie stwierdzić, że jestem dziekanem (ma na to dowód)
2. $PA \vdash p \iff \neg Prov(G(p))$ (tu jest dowód w PA)

a pewne niestety nie!

1. Jestem dziekanem (nie ma dowodu)
2. non $PA \vdash p$

Some further undecidable sentences...

Take a natural number, say:

1. m(0) = 1077

and let us represent it by a sum of the appropriate powers of 2 and exchange 2 for 3 in the next step:

1. $m(0) = 2^{2^{2+1}+1} + 2^{2^2+1} + 2^{2^2} + 2^2 + 1$
2. $m(0)' = 3^{3^{3+1}+1} + 3^{3^3+1} + 3^{3^3} + 3^3 + 1$

Let define now that $m(1) = 3^{3^{3+1}+1} + 3^{3^3+1} + 3^{3^3} + 3^3$. Finally, we exchange 3 for 4 and we subtract 1 etc. in order to obtain a sequence $m(n)$ for $n = 1, 2 \ldots$.

Theorem

Goodstein's theorem $\forall\ n\ lim_{n \to \infty} m(n) = 0$

Theorem

Downward Skolem-Lowenheim theorem *If a first-order theory T has an infinite model, it has a denumerable model.*

Theorem

*(**Upward Skolem-Loewenheim theorem**) If a first-order theory T has an infinite model with a cardinality α, than it has models of cardinalities $> \alpha$.*

Dowód.

The proof idea is as follows, taking $\mathcal{L}T$ and $T = \{F : A \models F\}$ alone, we:

- define $\mathcal{L}^{**} \cup \{c_i, i \in I\}$ and $T^{**} = T \cup \{\neg c_i \neq c_j, i \neq j\}$,
- consider finite subsets S of T^{**}, i.e. $S = \{F_1, \ldots, F_n\} \cup \{c_i \neq c_j, i, j \in I_{fin}\}$,
- we make use of compactness theorem which ensures the existence of a model for T^{**} if only each S has a model.

In order to find a model for each S, it enough to find a finite set of elements $a_i \neq a_j$, $i \neq j$ and take $\langle A, a_i \rangle$ as the model. □

Modal logic-satisfaction

Let $\mathcal{M} = \langle M, R, Val \rangle$ be a model for a modal logic system T, where R is an accessibility relation, Val – a set of true sentences in $\mathcal{M}$.

Definition

The satisfaction conditions are as follows:

1. $\mathcal{M}, u \models \phi \iff \phi \in Val,$
2. $\mathcal{M}, u \models \neg\phi \iff \mathcal{M}, u \not\models \phi$
3. $\mathcal{M}, u \models p \wedge q \iff \mathcal{M}, u \models p \wedge \mathcal{M}, u \models q$
4. $\mathcal{M}, u \models \Diamond\phi \iff \exists t(uRt \to \mathcal{M}, t \models \phi)$
5. $\mathcal{M}, u \models \Box\phi \iff \forall t(uRt \to \mathcal{M}, t \models \phi).$